On the p-divisibility of the sequence B_{lp^r}/lp^r.

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Let B_n $(n = 0, 1, 2, \ldots)$ be the usual nth Bernoulli number. For an irregular pair (p, l) we define

$$\Delta_{(p,l)} \equiv p^{-1} \left(\frac{B_{l+p-1}}{l+p-1} - \frac{B_l}{l} \right) \pmod{p}$$

with $0 \leq \Delta_{(p,l)} < p$. We call $\Delta_{(p,l)}$ *singular* when $\Delta_{(p,l)} = 0$. Note that no singular $\Delta_{(p,l)}$ has been found yet and that $p^2 \nmid B_l$ for $p < 12\,000\,000$; see [1] for these calculations.

The following theorems are simplified reformulations of [2, Theorem 8.1, p. 434] and [2, Corollary 8.2, p. 435]. See [2, Section 8, pp. 434] for further details.

Theorem 1 *Let $p \geq 5$ be a prime and l, r be positive integers where l is even and $0 < l < p$. There are the following cases.*

1. *If p is regular or (p, l) is not an irregular pair, then* $\text{ord}_p\,(B_{lp^r}/lp^r) = 0$.
2. *If (p, l) is an irregular pair and $p^2 \nmid B_{lp}/lp$, then* $\text{ord}_p\,(B_{lp^r}/lp^r) = 1$.
3. *If (p, l) is an irregular pair and $p^2 \mid B_{lp}/lp$, then* $\text{ord}_p\,(B_{lp^r}/lp^r) \geq 2$.

Note that no examples of case 3 of Theorem 1 are known. For a nonsingular $\Delta_{(p,l)}$, the following theorem gives a more precise result; see also [2, Section 4, pp. 415].

Theorem 2 *Let (p, l) be an irregular pair where $\Delta_{(p,l)} \neq 0$. The p-adic zeta function $\zeta_{p,l}$ associated with (p, l) has a unique simple zero $\chi_{(p,l)} \in \mathbb{Z}_p$. Then*

$$\text{ord}_p\,(B_{lp^r}/lp^r) = 1 + \text{ord}_p \left(\chi_{(p,l)} - l\frac{p^r - 1}{p - 1} \right).$$

Assume that only the first m $(m \geq 0)$ p-adic digits of $\chi_{(p,l)}$ are equal to l, $a_m \neq l$:

$$\chi_{(p,l)} = l + lp + lp^2 + \cdots + lp^{m-1} + a_m p^m + \cdots .$$

Then

$$\text{ord}_p\,(B_{lp^r}/lp^r) = 1 + \min(r, m).$$

The case that $\chi_{(p,l)} = l + \cdots$ is equivalent to $p^2 \mid B_{lp}/lp$ where no example is known. Moreover, it seems that the p-adic digits of $\chi_{(p,l)}$ are randomly distributed with no regularity.

References

[1] J. Buhler, R. Crandall, R. Ernvall, T. Metsänkylä, M. A. Shokrollahi. *Irregular primes and cyclotomic invariants to 12 million.* J. Symb. Comput. **31** (2001), no. 1–2, 89–96.

[2] B. C. Kellner. *On irregular prime power divisors of the Bernoulli numbers.* Math. Comp. **76** (2007), no. 257, 405–441.

Errata

[2], p. 434, Chapter 8, line 5 below

'In 1878 he computed a table of Bernoulli numbers $B_m^* = |B_{2m}|$ for $1 \leq m \leq 62$. On the basis of this table he conjectured[1] that $p \mid m$ implies $p \mid B_m^*$ for primes with $p - 1 \nmid 2m$; see [1].'